Last year, I was fortunate enough to take the Developing Mathematical Ideas (DMI) course on algebraic reasoning taught by Ruth Balf from the University of Washington, Seattle. DMI deepens participants’ mathematics knowledge while also modeling how to listen to students and ask the right questions at the right time. As Willis and others note, “When students are learning mathematics, they use their existing knowledge and understandings to make sense of new ideas” (2007, p. 22). DMI allows teachers to uncover that existing knowledge.

One of the primary messages is that children can and do construct their own meaning behind many complicated mathematical procedures and concepts. Taking a DMI course allows participants to return to their own classrooms and students with a fresh perspective.

The most fascinating piece of the course was examining how some higher-level algebraic concepts were viewed by very young children. I thought it impossible that children as young as first grade could construct advanced algebraic concepts. These are the very ideas that I struggled with myself in eighth and ninth grades. How could six- or seven-year-olds construct these notions without the same level of drill, instruction, and practice I had experienced?

A Young Student’s Concept of Zero

After a few sessions of the DMI course, I found myself growing more curious about how my own first-grade daughter would respond when presented with the notion of the value of zero and the concept of numbers less than zero. To set up our discussion, I used the game described in Number and Operations, Part 3: Reasoning Algebraically about Operations casebook (Schifter, Russell, and Bastable 2007, p. 86). The game requires students to move along a horizontal number line that has unlabeled marks to the left of zero and numbered marks to the right. The game requires the child to draw action cards labeled with a plus (+) sign or a minus (−) sign followed by a numeral. Players move a game piece to the correct location on the number line.

To further explore the concept of zero, I created a game where I drew a number line on a piece of paper and had my daughter move a game piece to indicate which numbers were positive and which were negative. We used a set of cards with numbers on them, and she had to place each card on the number line. This allowed her to see the concept of zero in a tangible way and understand that zero is neither positive nor negative.

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piece forward or backward on the number line to coincide with the action card they have chosen. Once we began playing the game, it did not take long for the concept of a value less than zero to arise. My daughter landed on zero and then drew a -3 card. She seemed stumped for a brief moment and then moved her finger three spaces to the left of zero. The following conversation between mother (M) and daughter (D) ensued:

D: [moving her token three spaces to the left of zero and counting aloud] Zero, zero, zero.
M: Wait a minute … zero, zero, zero? What does that mean? How can you have three zeroes?
D: I’m right here. I’m at nothing [pointing to the spot she landed on when she counted three spaces to the left of zero].
M: What do you mean, you’re at nothing? Your finger is somewhere. Are you telling me your finger is nowhere?
D: No, it’s right here [tapping the paper at the point three spots behind the zero].
M: So, you are somewhere. What would you call that place?
D: Nothing. It’s nothing. I can’t call it anything. 
M: But you just said you were here [pointing to daughter’s spot]. That is something, isn’t it?
D: Yeah … but … no … but … I don’t know. This is confusing for me.

**Zero is a hero**

I felt that there was more beneath the surface, so I made the decision to push her just a bit. I took a chance and actually gave her an idea of how to clear up her confusion. I asked her to give the spot she was referring to a name. She called it the zero cousin spot. She quickly and enthusiastically changed this name to the minus three spot.

M: Okay, let’s keep playing. Draw your next card [daughter draws a +1 card].
D: [moving her token one jump to the right] Now I’m on the minus two spot!
M: Why did you call it that?
D: ‘Cause this [as she pulls out a pencil and begins to write numbers to the left of the zero on the game board] would be the zero cousin minus one spot [pointing to the negative one position]; this would be the zero cousin minus two spot [pointing to the negative two position]; and this is the zero cousin minus three spot [pointing to the negative three position]. I added one to the minus three spot so I’m here at the minus two spot.”

It was clear from our discussion that my daughter was constructing her own ideas (pretty accurately) regarding numbers not only to the right but also to the left of the zero. Still, the whole notion of zero had me curious. In the DMI class, we had watched videos of children struggling with the contradiction of zero representing nothing but still standing for something. Because my daughter was in such good spirits, I decided to keep the discussion going.
and delve into her thinking on the concept of zero. We turned the game board over and used the same plus and minus action cards to display some basic mathematics problems. I put down the +4 card, and she pulled out four blocks. I then laid down the –4 card.

D: Hey … you can’t do that!
M: Why not?
D: Well, you’d just have nothing, that’s why.
M: Show me.
D: [picking up all the blocks and hiding them] See? Nothing!
M: I see! So, zero is nothing. You have nothing; it means nothing?
D: No, it means something. It means you don’t have anything.
M: So, is zero important, then?
D: Yeah! Zero is a hero! He takes the place of other numbers.
M: What do you mean by that?

At this point, she spent a significant amount of time explaining in detail how numbers such as 10, 20, 30, and 100 could not be made without zero. She proclaimed zero to be “the most important number ever.”

**Understanding numbers less than zero**
Having just touched on the concept of number less than zero, I pressed a bit further still:

M: Could you have something that’s less than zero?
D: No, you just stop at zero. It just ends and there is no more.
M: Hmmm … so, what if you had this?

I searched the card stack to find numbers to fit what I had in mind. I pulled out the +3 card and showed it to her. She pulled out three blocks. I then pulled the –4 card and showed it to her.

D: [after looking at the card for some time, takes all three blocks away] See, nothing. I can’t take anymore away, so it’s nothing. There is nothing less than zero.
M: Yeah, but over here [flipping the paper back to the previous game’s number line], you went behind the zero. Why couldn’t you do that here?
D: ’Cause over here I could still count behind the zero because you had these lines. I don’t have any more blocks or those lines to take away over here, so I can’t show you this number [pointing to the –1 position on the number line].
M: So, three minus four would put me here [pointing to the –1 position on the number line]?
D: Yep, that was the zero cousin minus one spot.
M: There is a spot here on the number line for that, but you can’t show it to me with the blocks?
D: Nope. I don’t have any left.
M: Would you ever want to have zero of anything?
D: Yeah! Zero school!
M: Would you ever want to have less than zero of anything?
D: I don’t think you can do that. Well … maybe if I gave you all of something I had but was still supposed to give you more, I’d have less than nothing. I don’t know how to do that.

It was wonderful to see her struggle with the notion of how to represent and name the number she counted as three less than zero. I enjoyed watching her develop, revise, and explain her numbering system to me. I was also impressed with how definitively she could explain and prove her answers. Throughout the activity and conversation, she showed a much higher understanding of zero and its purpose than I expected for her age. She even managed to show some beginning conceptual understanding of what it could mean to have something less than zero. Although her knowledge is by no means complete, she is beginning to realize that she may actually have to “owe” someone something to represent numbers less than zero.

![Photograph by Howard A. Wilcox III; all rights reserved](image)
Good Questions Show Us What Students Understand

How fascinating for me as both a parent and mathematics educator to watch these concepts as they were articulated for the first time. I was immediately reminded of one of the major talking points from the First Steps in Mathematics: Number course materials: “Good questions and tasks provoke students to show us what they know and understand” (Willis et al. 2007, p. 24). It is too bad that every mathematics teacher cannot implement this level of one-on-one questioning to really get to the heart of what each child they teach truly understands. As Tripp stated, “Students do misunderstand, but it is seldom because they cannot understand; most often it is because they understand something else” (1993, p. 88). I have a feeling that students who are written off as incapable of learning mathematics often fall into this category. They understand something; we educators have just not asked the right questions to figure out what they understand. Willis and others noted, “Understanding the mathematics helps educators make better professional judgments” (2007, p. 24). Reflecting on my conversation with my daughter, I would revise that idea to state, “Understanding the mathematics that children understand helps us make better professional judgments and, as a result, makes us better mathematics educators.”

What implications does this scenario hold for current classroom teachers or anyone developing and applying new mathematical ideas and working with students? Young children have logical thoughts and ideas about mathematical concepts to which they have never been exposed. For educators to go into any lesson assuming that they will have to provide or convey the bulk of the ideas to the students is a fallacy. As educators, we know that students come to the classroom with preconceptions about how the world works. It is our job to engage those ideas and uncover the logic and meaning behind what the students already know. If students’ initial understanding is not engaged, “they may fail to grasp the new concept or information, or they may learn what is taught for the purposes of passing a test but still cling to their initial preconceptions outside of the classroom arena” (Bransford and Donovan 2005, p. 1). Traditional mathematics instruction often attempts to override students’ reasoning processes and replace them with rules and procedures that disconnect the problem solving from meaning making. Bransford and Donovan point out that “informal strategy development and mathematical reasoning can serve as a foundation for learning more abstract mathematics … this link is not automatic” (pp. 218–19).

The technique of building on existing knowledge and the need to engage students’ preconceptions encompass the difference between true learning and simple memorization. Engaging and building on student preconceptions, then, poses an instructional challenge. Although there is surely no single best instructional approach, Bransford and Donovan (p. 223) suggest identifying certain features of instruction that support the following goals:

- “Allowing students to use their own informal problem-solving strategies, at least initially, and then guiding their mathematical thinking toward more effective strategies and advanced understandings.” Most of the standards-based curricula being used in many schools today allow for this to occur.
- “Encouraging math talk so that students can clarify their strategies to themselves and others, and compare the benefits and limitations of alternate approaches.” The Developing Mathematical Ideas (DMI) courses foster and encourage this level of mathematical discourse.
- “Designing instructional activities that can effectively bridge commonly held conceptions and targeted mathematical understandings.” The First Steps in Mathematics program (Willis et al. 2007) not only focuses on these conceptions but also allows teachers to identify and address the misconceptions constructed by students with high levels of competence.

I hope that my daughter’s teachers help her build and consolidate her previously constructed concepts. In order to understand any new concepts in depth and organize them in a way that makes sense to her and can be useful as she confronts increasingly difficult ideas and theories, she must have opportunities to build on concepts she has already constructed. The challenge for all teachers is to “provide sustained and then increasingly spaced opportunities to consolidate new understandings and procedures” (Bransford and Donovan 2005, p. 232).

An instructionally valid and appropriate lesson goes beyond the actual plan written on paper. It transcends the prepared blackline masters. It is more than the pages one plans to cover, the
books that will be shared, or the activities in which students will be engaged. A lesson will be instructionally appropriate and sound if the only things the teacher has written down are the high-level, probing questions she plans to ask her students about their understanding of the mathematical concepts during the lesson. Teachers may not be capable of asking every student each individual question (the way I was fortunate enough to do with my daughter), but the end product is far superior because the lesson is driven by students, their responses, and their innate understandings—not by the pages that must be covered or completed before a certain date. Making lessons learner-centered sends the message that the previously constructed knowledge not only is valued but also can be used to teach others in the class and move all participants (even the classroom teacher) forward in deepening their mathematical knowledge. Teachers must know their content well, understand the misconceptions students may bring to the concept, and be able to “build effectively on what learners bring to the classroom. They must pay close attention to individual students’ starting points and their progress on learning tasks. They must present students with ‘just-manageable difficulties’—challenging enough to maintain engagement and yet not so challenging as to lead to discouragement. They must find the strengths that will help students connect with the information being taught” (Bransford and Donovan 2005, p. 14).

References


