Realistic Mathematics Education

Introduction

This document highlights the importance of concept formation and problem solving in mathematics. An approach to teaching which promotes both of these is Realistic Mathematics Education (RME) which has been used successfully over many years in other countries. The basic principles underlying RME are outlined and evidence is given of improvement in students’ problem solving skills due to this approach.

The need for problem solving in school mathematics

There is widespread disquiet, both within and outside the educational community, about UK students’ mathematical achievement and, in particular, their ability to apply mathematics both in subsequent education and in employment. This was highlighted in The Smith report.

“In terms of usable skills, although GCSE grade C is the minimum societal expectation, evidence to the Inquiry suggests that employers are often less than happy about the mathematical abilities of recruits with GCSE, even when the grade obtained is at least a C.”

“The perception is that students are learning most of their mathematics in a vacuum, with little attention given to any sort of mathematical modelling, or to a range of problems set in real world contexts and using real data.”

In addition to the frequently expressed concerns about the lack of problem solving skills of students who pass GCSE Mathematics with grade C, there is evidence that some students who go on to study mathematics beyond GCSE do not have a sufficiently firm understanding in order to build on their knowledge successfully.

Ofsted’s 2006 report ‘Evaluating Mathematics Provision for 14 to 19 year olds’ found that factors which acted ‘against effective achievement, motivation and participation’ included the following.

“Teaching which presents mathematics as a collection of arbitrary rules and procedures, allied to a narrow range of learning activities in lessons which do not engage students in real mathematical thinking.”

and

“A narrow focus on meeting examination requirements by ‘teaching to the test’, so that although students are able to pass the examinations they are not able to apply their knowledge independently to new contexts, and they are not well prepared for further study.”
One of the Ofsted report’s recommendations in 2006 was as follows.

“The Department for Education and Skills, the Qualifications and Curriculum Authority and awarding bodies should ensure that the current revision of GCSE and A level mathematics results in examinations which encourage effective understanding and problem-solving, as well as competence in mathematical techniques.”

High ability students’ limited understanding of the concepts has an effect on their learning in the sciences as well as their continued progress in mathematics.

“In a recent research project we investigated students using their mathematics in the physical sciences (Wake and Hardy, 2005). This demonstrated how even high achieving students (in terms of their mathematics grades) have difficulty in seeing the common structure of laws and models of direct proportion such as Ohm’s Law, Hooke’s Law, Newton’s Second Law, the definition of density, and so on. Each was seen by the students as a new situation and their lack of insight into the common mathematical structure of each meant that it had to be learnt as a separate and new entity. Our investigations concluded that at no stage of current science or mathematics teaching is it likely that deep understanding of the underlying structure of these physical laws could emerge, and perhaps more worryingly it appears that there is currently no place where this is encouraged in either curriculum.”

In most UK classrooms, context may be used to introduce a topic but most learning of mathematical skills is done out of context. The skills are practised and only then applied to the solution of problems. This approach rarely appears to be questioned, despite its patent lack of success in equipping students with the desired conceptual understanding and problem solving skills. If UK students really are to become efficient problem solvers and functional mathematicians, it is important not only that we question current practice, but that other ways of working in the classroom are developed.

**Realistic Mathematics Education (RME)**

One possibility for equipping students with problem solving skills is provided by Realistic Mathematics Education. This was originally developed in The Netherlands.

The performance of The Netherlands in international comparisons of mathematical attainment has been consistently strong over recent years. The two major international comparative studies are PISA (Programme for International Student Assessment) and TIMSS (Trends in International Mathematics and Science Study). The former compares students’ mathematical problem solving abilities and is administered by the OECD, the latter measures purely mathematical attainment. The Netherlands usually scores well above average in both tests whereas England tends to be placed among the middle group.
Mathematics in The Netherlands uses a curriculum and theory of pedagogy developed over more than 30 years by the Freudenthal Institute, an internationally respected organisation. This curriculum, based on a philosophy called Realistic Mathematics Education (RME), is used by over 95% of schools and runs from kindergarten through to the later years of high school. One of the main features of this curriculum is that all the mathematics is developed through contexts, with a total commitment to enabling students to ‘make sense’ of the subject matter. This document outlines the main principles of RME and how it is radically different from the current practice in UK schools. It then considers the evidence from trialling RME in the UK.

Comparing RME with UK teaching of mathematics

A curriculum based on RME uses real contexts as both a route into mathematics and also as a means of developing students’ understanding. This contrasts sharply with the UK where, although context may at times be used as an introduction (and later as an application), the majority of work takes place in the abstract, with students ‘practising’ formal procedures. The majority of mathematics teaching in the UK can be summarised in the following diagram

Typical UK ‘Scheme’

- Context used for motivation and possibly for ‘access’.
- Context dropped by teacher.
- Techniques and procedures developed and practised out of context, working within the formal, abstract world of mathematics.
- Context re-introduced by teacher.
- ‘Applications’ / ‘Word problems’
  Used as further practice of techniques.

Introduction

Formalisation

Application

RME utilises a very different strategy for both the teaching of mathematics and for the development of problem solving skills. It provides students with a sequence of realistic problems, often based on the historical development of mathematical concepts. Through this device, students are led to reinvent the mathematics for themselves and gradually, over time, to use increasingly sophisticated methods. This is illustrated in the diagram below. It should be noted that there may well be more than 3 contexts utilised in the teaching of one topic.
The main features of RME

*Use of contexts*

When students work in context, rather than in the abstract, they are doing more than learning a particular type of mathematical technique. They are using mathematics to solve problems.

In RME, contexts are used not only to illustrate the applicability and relevance of mathematics in real world situations, but also as a source for the learning of mathematics itself. Contexts can be taken from the real world, from fiction or from an area of mathematics that students are already familiar with; the important thing is that they should be sufficiently real for students to be able to engage with them so that they are solving problems which make sense to them.

Students are encouraged to make sense of the context using their experiences, intuitions and common sense. They then stay in context, and remain at a sense-making level, while they develop mathematically. The word ‘realistic’ is used to emphasise that students are able to imagine the situation.
Experience shows that, through staying connected with the context, students are able to continue to make sense of what they are doing, and do not need to resort to memorising rules and procedures which have no meaning for them. ‘Mathematics’ and ‘context’ are not separated – to experience success in one implies success in the other.

The contexts used are extensively researched and differ significantly from those found in standard UK textbooks.

i) They regularly contain scope not only for different solution strategies, but also for different solutions.

ii) They allow students progressively to refine their thinking and to make connections as they deal with a progression of related contexts.

iii) They make much greater use of ‘real data’ and ‘unclean’ numbers. Hence estimation, a crucial ‘real world’ skill, plays a much more prominent part.

iv) There will often be more information than is required, and, occasionally, not enough. This is more akin to real world problems than the questions often encountered in mathematics books, where all the information in the question must be used in some way.

Use of ‘models’

RME provides a different view on how contexts should be chosen, and also on how these can then be used to support mathematical development. The use of ‘models’ is crucial here.

A model emerges from a context. Initially it may be little more than a representation, for example a picture, suggested by the context. Later, however, these models become more sophisticated mathematical tools such as the number line, ratio tables, etc.

Models bridge the gap between the informal and the formal and so teachers feel less pressure to replace students’ informal knowledge with formal procedures. Models also allow students to work at differing levels of abstraction, so that those who have difficulty with more formal notions can still make progress and will still have strategies for solving problems.

An important part of a student’s mathematical development is the recognition that the same model can be used in a variety of situations and to structure solutions to many kinds of problems. Further exemplification of this can be found in Appendix 1.

The mathematics levels in the 1999 National Curriculum for England characterise students’ development by means of the procedures which they are able to perform. However, in RME, teachers use a richer range of descriptors to gauge students’ progress. This includes observing students’ use of models, insights and reflection as well as mathematical landmarks and procedures.
**Multiple strategies**

One aspect of being ‘functional’ in mathematics is being able to choose the most appropriate strategy to solve a problem, rather than always relying on one strategy or algorithm. The contexts in RME are chosen to elicit many different strategies and students are constantly encouraged to reflect on these and refine them. Lessons will involve comparison and evaluation of different student strategies; a fundamental tenet of RME is to build sophistication into student-generated procedures rather than for teachers to impose a ‘standard method’ or algorithm. Further exemplification of this is given in Appendix 1.

RME encourages the development of more formal methods from students’ informal methods. However, it also allows students to continue to be able to use informal methods, where appropriate, rather than relying on being taught a method of solution that fits a certain type of problem. This results in students who are more able to solve unfamiliar problems; there is an example of this in Appendix 2.

The types of responses seen in Appendix 2 strongly suggest that introducing formal mathematical knowledge at too early a stage in a student’s development will replace any more informal ideas and leave the learner with no knowledge at all once a rule is forgotten.

**The Manchester Metropolitan University Key Stage 3 pilot**

Several countries have engaged The Freudenthal Institute (FI) to advise on curriculum matters. In the United States, a major collaboration between FI and The University of Wisconsin has produced a Middle school curriculum which has been adopted by a considerable number of school districts. This curriculum, known as Mathematics in Context (MiC) produces impressive student achievement; this is described in the 2003 book ‘Standards-based School Mathematics Curricula’, which can be previewed on Google Books.

In 2003, staff at Manchester Metropolitan University (MMU) purchased the MiC materials from the US and trialled them in a local school. This led, in 2004, to a major curriculum development project, funded by the Gatsby foundation. In the next 4 years, more than 3000 students have been involved in following the MiC curriculum, initially in Manchester but now also in other schools in the UK.

The progress of these students has been very encouraging. It is significant that, of the schools involved at this stage, most have volunteered to be part of a project developing RME materials for Key Stage 4 students.
**Problem solving**

As part of the Manchester Key Stage 3 pilot, a problem solving test was set at the end of each year. This involved project students and also ‘control’ groups, matched by Key Stage 2 attainment; each of the four groups contained 100 students.

The table below shows the proportion of year 7 students getting each question correct. The test can be seen at [http://www.partnership.mmu.ac.uk/cme/DMtC/Updates/RepAnnual2006App1.pdf](http://www.partnership.mmu.ac.uk/cme/DMtC/Updates/RepAnnual2006App1.pdf)

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Students’ Key Stage 2 Level</th>
<th>3 to low 4 (0-25&lt;sup&gt;th&lt;/sup&gt; percentile ability range)</th>
<th>high 4 to 5 (35&lt;sup&gt;th&lt;/sup&gt; - 75&lt;sup&gt;th&lt;/sup&gt; percentile ability range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Project (N=100)</td>
<td>Control (N=100)</td>
<td>Project (N=100)</td>
</tr>
<tr>
<td>1(a)</td>
<td>23%</td>
<td>12%</td>
<td>52%</td>
</tr>
<tr>
<td>1(b)</td>
<td>10%</td>
<td>2%</td>
<td>23%</td>
</tr>
<tr>
<td>2(a)</td>
<td>42%</td>
<td>7%</td>
<td>65%</td>
</tr>
<tr>
<td>2(b)</td>
<td>54%</td>
<td>12%</td>
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<tr>
<td>3(a)</td>
<td>50%</td>
<td>35%</td>
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</tr>
<tr>
<td>3(b)</td>
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<td>3(D)</td>
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<td>64%</td>
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<tr>
<td><strong>Average</strong></td>
<td><strong>36</strong></td>
<td><strong>17</strong></td>
<td><strong>55</strong></td>
</tr>
</tbody>
</table>

These results show major differences between the RME (project) students and the other (control group) students, particularly with those of middle and lower ability.

The RME students not only significantly outscored control students in terms of correct answers, but also showed much more sophistication in their choice of strategies and a willingness to ‘have a go’ when unsure how to proceed.

Similar differences were seen between RME students and the control groups in year 8. In addition to this, there were significant differences in attitudes towards mathematics between the two groups of students.

- 63% of RME students said that the maths they are studying will help them with their adult life after school compared to 30% of the control students.
- 75% of RME students said that the maths they do in school is useful for situations outside schools compared to 34% of the control students.
- 71% percent of the RME students said that maths lessons are useful compared to 30% of the control students.
Whenever examples of RME students’ work are presented, observers are struck by the range of strategies employed by the students, and by the ‘functional’ nature of their problem solving skills.

Examples of students’ work on question 1b can be found in Appendix 2.

**Key Stage 3 National Curriculum tests**

While the improvement in students’ problem solving skills is easily seen, it raises the question of whether it has been achieved at the expense of their ability to perform routine techniques and achieve their potential in national tests.

In 2007, the first cohort of students (over 300) took the National Curriculum Key Stage 3 tests after three years of using RME. Their results were compared with those of students from two schools from the same area, matched for GCSE attainment, catchments and the general quality of the teaching in the schools.

The added value from Key Stage 2 to Key Stage 3 for the project schools was slightly higher than in the control schools; this is despite the fact that the Mathematics in Context teaching materials had not been designed for our Key Stage 3 curriculum and probably only covered in the region of 80% of the National Curriculum at this level.

It should be noted that evidence from similar projects of this nature, where an innovative curriculum is introduced, suggests that the use of standardised tests matched to the original curriculum tends to underestimate the gains in understanding made by students. For example, Schoenfeld describes a study in which students’ performance on a problem-solving test is compared with their performance on a standard test of mathematical techniques. 95% of students who passed the problem-solving test also passed the standard test but only 78% of students who passed the standard test were able to pass the problem-solving test.

The emphasis of the programme which the project students had followed was on problem solving in context, rather than on teaching the specific techniques listed in the 1999 National Curriculum. However, they were still successful in national tests which compared them to students who had followed a more traditional programme.

The new National Curriculum (for teaching from September 2008 at Key Stage 3 and 2010 at Key Stage 4) stresses the importance of process skills and problem solving in mathematics. The philosophy of RME is well-aligned to this emphasis and, hence, an approach to teaching and learning based on RME will help schools to implement the new curriculum successfully.

**Moving on to Key Stage 4**

In response to concerns about mathematics at Key Stage 4 and the new Functional Mathematics agenda, a collaboration between MMU and The Freudenthal Institute was established in 2007 to produce materials based on RME for the new Foundation Level GCSE. Materials have so far been produced for Year 10 and these have been trialled in 7 Manchester schools. A further 15 schools have agreed to be involved in the trial from September 2008.
While this is still in its early stages, teachers are reporting very similar outcomes to those experienced at Key Stage 3. Many of the students involved have previously experienced little success in mathematics, yet they are generally showing signs of being well motivated by the new materials and of enjoying the subject. Assessments done to date show some noteworthy gains compared to equivalent groups, with students more willing to engage with problem solving and to try to ‘make sense’ of their mathematics.

What teachers say about RME
It is clear that working with RME can be a challenging experience for teachers, both in terms of their beliefs and their classroom practice. The materials are not simply books for students to work from, but problems which demand discussion and reflection, both in small groups and as a whole class. In this respect, RME not only provides scaffolding for student learning but also for teachers’ professional development. Working with RME, and attending CPD sessions has led teachers to re-evaluate their classroom practices and their beliefs about what constitutes mathematical development for students. One teacher on the project commented after two years of working with MiC materials that “I saw MiC as the book to start with, now I see MiC as a way of teaching.”

Sue Hough, who has worked with the MMU project for several years, says: “These materials foster a purpose in pupils which I have failed to generate in 15 years of teaching. They give the pupils an ownership over their mathematics, an ability to make sense of their methods and therefore the confidence to contribute with authority to a class discussion. One boy in my bottom set who had spent two years’ worth of lessons turning round, even when moved to the back of the room, awoke one day from his mathematical slumber to voluntarily share a mathematical idea which none of us had considered. That to me was an indicator of real progress.”

Concluding remarks
The evidence from other countries, supported by small scale trialling of the RME approach in the UK, is that it leads to greater student engagement with mathematics, increased understanding of the underlying concepts and an improvement in problem solving skills.

The indications are that adopting this approach to teaching and learning mathematics in England would result in more students being genuinely functional in their ability to use mathematics.
Appendix 1.

The RME approach to the teaching of fractions.

In this section, the topic of fractions is used to exemplify significant differences between RME and typical teaching in UK schools.

From 1999 to 2006, QCA published Implications for Teaching and Learning based on student performance in National Curriculum Tests. It is evident from this feedback that fractions are a difficult topic for many students at Key Stage 3. However, fractions are included in the curriculum from Year 2. Equivalence of fractions and common denominators are worked on in each subsequent year until Year 9 when students should be able to ‘recognise and use the equivalence of fractions’. So, on the one hand, students continue, year on year, to experience difficulty with fractions and the notion of equivalence, yet, on the other hand, this notion is being worked on from Year 2 all the way through to Year 9 (and for many students beyond)!

So what is different about the RME approach?
At an early stage students work with a context involving baguettes called ‘submarine sandwiches’. They are encouraged to draw diagrams to show how they would share them out with groups of people. This context is chosen as the rectangular shape of the baguette helps students to see the bar as a possible model for working with fractions.

![Fig. 1](image)

This use of a rectangle to represent the whole is also encouraged by a variety of other contexts such as cans of coconut milk, cutting up (conveniently rectangular) food, and then shading a (rectangular) meter to represent the occupancy of a car park. Initially, the fraction bar is a simplified picture of the problem. However, as students consider questions such as whether it is possible to pour Can A of coconut milk (1/3 full) into Can B (3/4 full), the model becomes a tool for comparing different fractions. Although it is still closely linked to the context; many students draw rectangles instead of cans at this stage.
The model of the bar for a fraction is extended further for addition and subtraction of fractions. In Fig. 3 one of the cartons is 3/10 apple juice, and the other 1/4, and a bar of length 40 is used to compare the amounts.

It is through this progressive formalisation of models (Fig. 4), from picture of a context to more abstract mathematical diagrams, that students make progress towards the formal notions of equivalence and the use of common denominators. However, this happens in a way which allows students to stay close to the ‘reality’ of the situation, and to return to more informal, primitive strategies as the need arises.
This progression from informal to formal, and the consideration of different strategies, has led to some impressive results. These are looked at in more detail in Appendix 2.
Appendix 2

Students’ approach to solving problems

As described earlier, part of the Manchester pilot involved project and control students sitting a problem solving test. As well as the actual marks, given in the table on page 7, this provided information about the ways the two groups of students tackled the problems.

Far more project students attempted to ‘make sense’ of the problems, and this was particularly true of lower ability students, many of whom did not get correct answers. For example, question 1 (b) was as follows.

*Find the area of the shape shown below.*
*Show carefully how you worked it out.*

![Shape Diagram]

This was included in the problem solving test as no Year 7 students had previously met the shape and so no formula was known. Consequently, just performing a calculation with the numbers in the question did not give the correct answer, unless a student had thought about how the area might be found.

Strategies classified as ‘sense-making’ were
- drawing cm squares and counting
- splitting into rectangle and triangles
- moving a triangle to create a rectangle

By contrast, purely numerical strategies included
- adding all the numbers given
- multiplying the numbers
- averaging the numbers in some way

74% of project students in the lower two ability bands used a ‘sense-making’ strategy. Only 32% of the corresponding control students were able to do this. Indeed, the vast majority of control students adopted a purely numerical method. This is despite the fact that all will have counted squares when initially introduced to area. Two incorrect responses from two lower ability students (one project and one control) are shown below to illustrate this point.
The project student, whose work is shown above, would have got the right answer if it had not been for a numerical slip (using 6 instead of 5 in the final calculation). By contrast, the control student whose work is shown below is just trying to do something with all the numbers given in the question. This is despite the fact that this makes no sense.
Lower ability control student adopting a purely numerical approach.

References
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4. Evaluating Mathematics Provision for 14 to 19 year olds, Ofsted, 2006
5. Evaluating Mathematics Provision for 14 to 19 year olds, Ofsted, 2006